A PERIODIC PROBLEM OF HEAT CONDUCTION IN A HOLLOW INFINITE CYLINDER

R. S. Minasyan

UDC 536.2.01

The problem of radial distribution of temperature in an infinite hollow cylinder is solved in the presence of heat exchange with the surrounding medium in the case in which the heat exchange coefficients are periodic functions of time.

In practice, among various cases of heat exchange between a body and the surrounding medium, one comes quite frequently across a case in which the heat exchange coefficients vary in the course of time. An explicit solution is given below to the problem of radial distribution of temperature in an infinite hollow cylinder of circular cross section, in the presence, on the outer and inner surfaces, of heat exchange with the surrounding medium when the heat-exchange coefficients are periodic functions of time. Such problems arise, for example, in the case of periodically alternating vaporization and steam condensation on surfaces or in a periodic blowing or injection into a coolant nozzle, etc. It is assumed that heat sources within the body vary periodically. In this case the temperature distribution function u(r, t) is a solution of the following differential equation [1]:

$$\frac{\partial u}{\partial t}=a\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right)+\frac{1}{c\rho}P(r,\ t) \tag{1}$$

with the boundary conditions

$$\begin{aligned}-\frac{\partial u}{\partial r}\Big|_{r=R_1}&=h_1(t)\left[S_1(t)-u(R_1,\ t)\right],\\ \frac{\partial u}{\partial r}\Big|_{r=R_2}&=h_2(t)\left[S_2(t)-u(R_2,\ t)\right],\end{aligned} \tag{2}$$

where $a=\lambda/c\rho$ is the coefficient of temperature conductivity; λ is the heat-conduction coefficient; ρ is the body density; c is the heat capacity coefficient; P(r, t) is the rate of heat release; $h_1(t)$, $S_1(t)$, $h_2(t)$, $S_2(t)$ are the heat-exchange coefficients and the temperature of the surrounding medium on the surfaces $r=R_1$ and $r=R_2$ respectively. As regards the functions $h_1(t)$, $h_2(t)$, $S_1(t)$, $S_2(t)$ and P(r, t) one assumes that they are of bounded variation in the interval $(0,\ \vartheta)$ where ϑ denotes the period of the function u(r, t).

If to Eq. (1) one applies the finite complex Fourier transformation with respect to time t then the following differential equation is obtained for the image of the function:

$$F_k''(r)+\frac{1}{r}F_k'(r)+\frac{i\gamma_k}{a}F_k(r)=-\frac{1}{\lambda}P_k(r), \tag{3}$$

where

$$F_k(r)=\int_0^{\vartheta}u(r,\ t)\,e^{i\gamma_k t}dt,\quad P_k(r)=\int_0^{\vartheta}P(r,\ t)\,e^{i\gamma_k t}dt;\quad \gamma_k=\frac{2k\pi}{\vartheta}.$$

Solving Eq. (3) one obtains

$$F_k(r)=\frac{1}{Q_k(R_1,\ R_2)}\left\{Q_k(r,\ R_2)\left[M_k^{(1)}-\frac{1}{\lambda}\int_{R_1}^{r}r_1P_k(r_1)Q_k(R_1,\ r_1)\,dr_1\right]+Q_k(R_1,\ r)\left[M_k^{(2)}-\frac{1}{\lambda}\int_{r}^{R_2}r_1P_k(r_1)Q_k(r_1,\ R_2)\,dr_1\right]\right\}, \tag{4}$$

where, for brevity, the notation

Institute of Mathematics and Mechanics, Academy of Sciences of the Armenian SSR, Erevan. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 17, No. 5, pp. 880-891, November, 1969. Original article submitted December 25, 1968.

$$Q_k(x, y) = Y_0(\sqrt{i}\,\mu_k x)\,J_0(\sqrt{i}\,\mu_k y) - Y_0(\sqrt{i}\,\mu_k y)\,J_0(\sqrt{i}\,\mu_k x),$$
$$\mu_k = \sqrt{\frac{\gamma_k}{a}}.$$

is introduced. In the above $J_0(x)$, $Y_0(x)$ are Bessel functions of zero order of the first or second kind respectively; $M_k^{(1)}$ and $M_k^{(2)}$ are integration constants.

For $k = 0$ one has

$$F_0(r) = -\frac{1}{\ln R_2 - \ln R_1}\left\{\ln\frac{r}{R_1}\left[M_0^{(2)} + \frac{1}{\lambda}\int_r^{R_2} r_1 P_0(r_1)\ln\frac{R_2}{r_1}\,dr_1\right] + \ln\frac{R_2}{r}\left[M_0^{(1)} + \frac{1}{\lambda}\int_{R_1}^{r} r_1 P_0(r_1)\ln\frac{r_1}{R_1}\,dr_1\right]\right\}. \quad (5)$$

In accordance with the inversion formula [2], the formula which gives the original in terms of the image of the function is given by the series

$$u(r, t) = \frac{1}{\vartheta}\sum_{k=-\infty}^{\infty} F_k(r)\,e^{-i\gamma_k t}. \quad (6)$$

Prior to determining the constants $M_k^{(1)}$ and $M_k^{(2)}$ the boundary conditions (2) are modified and rewritten as

$$-\left.\frac{\partial u}{\partial r}\right|_{r=R_1} + h_1^* u(R_1, t) = h_1(t)\,S_1(t) - \left[h_1(t) - h_1^*\right] u(R_1, t);$$
$$\left.\frac{\partial u}{\partial r}\right|_{r=R_2} + h_2^* u(R_2, t) = h_2(t)\,S_2(t) - \left[h_2(t) - h_2^*\right] u(R_2, t), \quad (7)$$

where

$$h_l^* = \frac{1}{\vartheta}\int_0^{\vartheta} h_l(t)\,dt.$$

Applying the finite complex Fourier transformation to the conditions (7) the following totality of infinite systems of linear algebraic equations is obtained for the determination of $M_k^{(1)}$ and $M_k^{(2)}$:

$$M_k^{(1)} = \frac{1}{W_k^{(1)}}\left\{Q_k(R_1, R_2)\left[-\sum_{j=-\infty}^{\infty}{}' M_j^{(1)} C_{k-j}^{(1)} + \int_0^{\vartheta} h_1(t)\,S_1(t)\,e^{i\gamma_k t}\,dt\right] - \frac{1}{R_1}\left[M_k^{(2)} - \frac{1}{\lambda}\int_{R_1}^{R_2} rP_k(r)\,Q_k(r, R_2)\,dr\right]\right\},$$
$$M_k^{(2)} = \frac{1}{W_k^{(2)}}\left\{Q_k(R_1, R_2)\left[-\sum_{j=-\infty}^{\infty}{}' M_j^{(2)} C_{k-j}^{(2)} + \int_0^{\vartheta} h_2(t)\,S_2(t)\,e^{i\gamma_k t}\,dt\right] - \frac{1}{R_2}\left[M_k^{(1)} - \frac{1}{\lambda}\int_{R_1}^{R_2} rP_k(r)\,Q_k(R_1, r)\,dr\right]\right\}. \quad (8)$$

In the above

$$W_k^{(1)} = \sqrt{i}\,\mu_k\left[Y_0(\sqrt{i}\,\mu_k R_2)\,J_0'(\sqrt{i}\,\mu_k R_1) - J_0(\sqrt{i}\,\mu_k R_2)\,Y_0'(\sqrt{i}\,\mu_k R_1)\right] + h_1^* Q_k(R_1, R_2),$$
$$W_k^{(2)} = \sqrt{i}\,\mu_k\left[Y_0(\sqrt{i}\,\mu_k R_1)\,J_0'(\sqrt{i}\,\mu_k R_2) - J_0(\sqrt{i}\,\mu_k R_1)\,Y_0'(\sqrt{i}\,\mu_k R_2)\right] + h_2^* Q_k(R_1, R_2), \quad (9)$$
$$C_k^{(l)} = \frac{1}{\vartheta}\int_0^{\vartheta} h_1(t)\,e^{i\gamma_k t}\,dt, \quad l = 1;\ 2,$$

in which the dash under the summation sign indicates that the term corresponding to $j = k$ is omitted.

For $k = 0$ one obtains

$$M_0^{(1)} = \frac{1}{W_0^{(1)}}\left\{\ln\frac{R_2}{R_1}\left[-\sum_{j=-\infty}^{\infty}{}' M_j^{(1)} C_{-j}^{(1)} + \int_0^{\vartheta} h_1(t)\,S_1(t)\,dt\right] + \frac{1}{R_1}\left[M_0^{(2)} + \frac{1}{\lambda}\int_{R_1}^{R_2} rP_0(r)\ln\frac{R_2}{r}\,dr\right]\right\},$$
$$M_0^{(2)} = \frac{1}{W_0^{(2)}}\left\{\ln\frac{R_2}{R_1}\left[-\sum_{j=-\infty}^{\infty}{}' M_j^{(2)} C_{-j}^{(2)} + \int_0^{\vartheta} h_2(t)\,S_2(t)\,dt\right] + \frac{1}{R_2}\left[M_0^{(1)} + \frac{1}{\lambda}\int_{R_1}^{R_2} rP_0(r)\ln\frac{r}{R_1}\,dr\right]\right\}, \quad (10)$$

where

$$W_0^{(1)} = \frac{1}{R_1} + h_1^* \ln \frac{R_2}{R_1}, \quad W_0^{(2)} = \frac{1}{R_2} + h_2^* \ln \frac{R_2}{R_1}.$$

To investigate the system (8) and (10) one separates first the real and imaginary parts. One introduces the notation

$$M_k^{(l)} = (m_k^{(l)} + in_k^{(l)}) \, |k|^{-\frac{5}{4}}, \; M_0^{(l)} = m_0^{(l)}, \quad l = 1;\ 2, \tag{11}$$

and one obtains from (8) the following relations for the new unknowns $m_k^{(1)}$ and $n_k^{(1)}$:

$$m_k^{(1)} = -\frac{1}{2g_k^{(1)}}\Big\{m_0^{(1)} k^{\frac{5}{4}} (A_k^{(1)}L_k^{(1)} + D_k^{(1)}N_k^{(1)}) + k^{\frac{5}{4}} \sum_{j=1}^{\infty}{}' j^{-\frac{5}{4}} \big[m_j^{(1)} ((A_{k-j}^{(1)} + A_{k+j}^{(1)}) L_k^{(1)} + (D_{k-j}^{(1)} + D_{k+j}^{(1)}) N_k^{(1)})$$

$$- n_j^{(1)} ((A_{k+j}^{(1)} - A_{k-j}^{(1)}) N_k^{(1)} + (D_{k-j}^{(1)} - D_{k+j}^{(1)}) L_k^{(1)})\big] + \frac{2}{R_1} \big[m_k^{(2)} (\zeta_k^{(1)} + h_1^*\eta_k (R_1, R_2)) + n_k^{(2)} (\psi_k^{(1)} + h_1^*\xi_k (R_1, R_2))\big]\Big\} + p_k^{(1)},$$

$$n_k^{(1)} = -\frac{1}{2g_k^{(1)}}\Big\{m_0^{(1)} k^{\frac{5}{4}} (D_k^{(1)}L_k^{(1)} - A_k^{(1)}N_k^{(1)}) + k^{\frac{5}{4}} \sum_{j=1}^{\infty}{}' j^{-\frac{5}{4}} \big[m_j^{(1)} ((D_{k+j}^{(1)} + D_{k-j}^{(1)}) L_k^{(1)} - (A_{k+j}^{(1)} + A_{k-j}^{(1)}) N_k^{(1)})$$

$$+ n_j^{(1)} ((A_{k-j}^{(1)} - A_{k+j}^{(1)}) L_k^{(1)} - (D_{k+j}^{(1)} - D_{k-j}^{(1)}) N_k^{(1)})\big]$$

$$+ \frac{2}{R_1} \big[m_k^{(2)} (\psi_k^{(1)} + h_1^*\xi_k (R_1, R_2)) - n_k^{(2)} (\zeta_k^{(1)} + h_1^*\eta_k (R_1, R_2))\big]\Big\} + q_k^{(1)}. \tag{12}$$

In the above the following notation was introduced:

$$A_k^{(l)} = \frac{2}{\vartheta} \int_0^{\vartheta} h_l(t) \cos \gamma_k t dt, \quad D_k^{(l)} = \frac{2}{\vartheta} \int_0^{\vartheta} h_l(t) \sin \gamma_k t dt, \qquad l = 1;\ 2,$$

$$g_k^{(l)} = [\zeta_k^{(l)} + h_l^*\eta_k (R_1, R_2)]^2 + [\psi_k^{(l)} + h_l^*\xi_k (R_1, R_2)]^2,$$

$$L_k^{(l)} = \zeta_k^{(l)}\eta_k (R_1, R_2) + \psi_k^{(l)}\xi_k (R_1, R_2) + h_l^* [\eta_k^2 (R_1, R_2) + \xi_k^2 (R_1, R_2)],$$

$$\eta_k (x, y) = \operatorname{ker} (\mu_k x) \operatorname{ber} (\mu_k y) - \operatorname{kei} (\mu_k x) \operatorname{bei} (\mu_k y) - \operatorname{ber} (\mu_k x) \operatorname{ker} (\mu_k y) + \operatorname{bei} (\mu_k x) \operatorname{kei} (\mu_k y),$$

$$\xi_k (x, y) = \operatorname{kei} (\mu_k x) \operatorname{ber} (\mu_k y) + \operatorname{ker} (\mu_k x) \operatorname{bei} (\mu_k y) - \operatorname{bei} (\mu_k x) \operatorname{ker} (\mu_k y) - \operatorname{ber} (\mu_k x) \operatorname{kei} (\mu_k y),$$

$$\zeta_k^{(1)} = \mu_k [\operatorname{kei}' (\mu_k R_1) \operatorname{bei} (\mu_k R_2) - \operatorname{ker}' (\mu_k R_1) \operatorname{ber} (\mu_k R_2) - \operatorname{bei}' (\mu_k R_1) \operatorname{kei} (\mu_k R_2) + \operatorname{ber}' (\mu_k R_1) \operatorname{ker} (\mu_k R_2)],$$

$$\psi_k^{(1)} = \mu_k [\operatorname{ber}' (\mu_k R_1) \operatorname{kei} (\mu_k R_2) + \operatorname{bei}' (\mu_k R_1) \operatorname{ker} (\mu_k R_2) - \operatorname{kei}' (\mu_k R_1) \operatorname{ber} (\mu_k R_2) - \operatorname{ker}' (\mu_k R_1) \operatorname{bei} (\mu_k R_2)],$$

$$N_k^{(l)} = \zeta_k^{(l)}\xi_k (R_1, R_2) - \psi_k^{(l)}\eta_k (R_1, R_2), \tag{13}$$

$$p_k^{(1)} = \frac{k^{\frac{5}{4}}}{g_k^{(1)}} \Big\{\frac{1}{\lambda R_1} \int_{R_1}^{R_2} \int_0^{\vartheta} rP (r, t) \big[(\zeta_k^{(1)} + h_1^*\eta_k (R_1, R_2)) (\eta_k (r, R_2) \cos \gamma_k t$$

$$+ \xi_k (r, R_2) \sin \gamma_k t) + (\psi_k^{(1)} + h_1^*\xi_k (R_1, R_2)) (\xi_k (r, R_2) \cos \gamma_k t$$

$$- \eta_k (r, R_2) \sin \gamma_k t)\big] dt dr + \int_0^{\vartheta} h_1 (t) S_1 (t) (L_k^{(1)} \cos \gamma_k t + N_k^{(1)} \sin \gamma_k t) dt\Big\},$$

$$q_k^{(1)} = \frac{k^{\frac{5}{4}}}{g_k^{(1)}} \Big\{\frac{1}{\lambda R_1} \int_{R_1}^{R_2} \int_0^{\vartheta} rP (r, t) \big[(\zeta_k^{(1)} + h_1^*\eta_k(R_1, R_2)) (\eta_k (r, R_2) \sin \gamma_k t$$

$$- \xi_k (r, R_2) \cos \gamma_k t) + (\psi_k^{(1)} + h_1^*\xi_k (R_1, R_2)) (\eta_k (r, R_2) \cos \gamma_k t$$

$$+ \xi_k (r, R_2) \sin \gamma_k t)\big] dt dr + \int_0^{\vartheta} h_1 (t) S_1 (t) (L_k^{(1)} \sin \gamma_k t - N_k^{(1)} \cos \gamma_k t) dt \Big\},$$

where ber(x), bei(x), ker(x) and kei(x) are Thomson functions of zero order of the first and second kind respectively which are the real and imaginary parts of a Bessel function of the complex argument.

The values of the unknowns $m_k^{(2)}$ and $n_k^{(2)}$ are given by the following systems of equations:

$$m_k^{(2)} = -\frac{1}{2g_k^{(2)}}\Big\{m_0^{(2)} k^{\frac{5}{4}} (A_k^{(2)}L_k^{(2)} + D_k^{(2)}N_k^{(2)}) + k^{\frac{5}{4}} \sum_{j=1}^{\infty}{}' j^{-\frac{5}{4}} \big[m_j^{(2)} ((A_{k+j}^{(2)} + A_{k-j}^{(2)}) L_k^{(2)} + (D_{k+j}^{(2)} + D_{k-j}^{(2)}) N_k^{(2)})$$

$$+n_j^{(2)}\left((D_{k+j}^{(2)}-D_{k-j}^{(2)})L_k^{(2)}-(A_{k+j}^{(2)}-A_{k-j}^{(2)})N_k^{(2)}\right)\Big]+\frac{2}{R_2}\left[m_k^{(1)}\left(\zeta_k^{(2)}+h_2^*\eta_k(R_1,R_2)\right)-n_k^{(1)}\left(\psi_k^{(2)}+h_2^*\xi_k(R_1,R_2)\right)\right]\Big\}+p_k^{(2)},$$

$$n_k^{(2)}=-\frac{1}{2g_k^{(2)}}\Big\{m_0^{(2)}k^{-\frac{5}{4}}(D_k^{(2)}L_k^{(2)}-A_k^{(2)}N_k^{(2)})+k^{\frac{5}{4}}\sum_{j=1}^{\infty}{}'j^{-\frac{5}{4}}\Big[m_j^{(2)}\left((D_{k+j}^{(2)}+D_{k-j}^{(2)})L_k^{(2)}-(A_{k+j}^{(2)}+A_{k-j}^{(2)})N_k^{(2)}\right) \tag{14}$$

$$-n_j^{(2)}\left((A_{k+j}^{(2)}-A_{k-j}^{(2)})L_k^{(2)}+(D_{k+j}^{(2)}-D_{k-j}^{(2)})N_k^{(2)}\right)\Big]+\frac{2}{R_2}\left[n_k^{(1)}\left(\zeta_k^{(2)}+h_2^*\eta_k(R_1,R_2)\right)+m_k^{(1)}\left(\psi_k^{(2)}+h_2^*\xi_k(R_1,R_2)\right)\right]\Big\}+q_k^{(2)},$$

where

$$\zeta_k^{(2)}=\mu_k\left[\mathrm{ber}'(\mu_kR_2)\ker(\mu_kR_1)-\mathrm{bei}'(\mu_kR_2)\mathrm{kei}(\mu_kR_1)-\ker'(\mu_kR_2)\mathrm{ber}(\mu_kR_1)+\mathrm{kei}'(\mu_kR_2)\mathrm{bei}(\mu_kR_1)\right],$$

$$\psi_k^{(2)}=\mu_k\left[\mathrm{ber}'(\mu_kR_2)\mathrm{kei}(\mu_kR_1)+\mathrm{bei}'(\mu_kR_2)\ker(\mu_kR_1)-\ker'(\mu_kR_2)\mathrm{bei}(\mu_kR_1)-\mathrm{kei}'(\mu_kR_2)\mathrm{ber}(\mu_kR_1)\right],$$

$$p_k^{(2)}=\frac{k^{-\frac{5}{4}}}{g_k^{(2)}}\Big\{\frac{1}{\lambda R_2}\int_{R_1}^{R_2}\int_0^{\vartheta}rP(r,t)\left[\left(\zeta_k^{(2)}+h_2^*\eta_k(R_1,R_2)\right)\left(\eta_k(R_1,r)\cos\gamma_kt+\xi_k(R_1,r)\sin\gamma_kt\right)\right.$$

$$\left.-\left(\psi_k^{(2)}+h_2^*\xi_k(R_1,R_2)\right)\left(\eta_k(R_1,r)\sin\gamma_kt-\xi_k(R_1,r)\cos\gamma_kt\right)\right]dtdr+\int_0^{\vartheta}h_2(t)S_2(t)(L_k^{(2)}\cos\gamma_kt+N_k^{(2)}\sin\gamma_kt)\,dt\Big\}, \tag{15}$$

$$q_k^{(2)}=\frac{k^{-\frac{5}{4}}}{g_k^{(2)}}\Big\{\frac{1}{\lambda R_2}\int_{R_1}^{R_2}\int_0^{\vartheta}rP(r,t)\left[\left(\zeta_k^{(2)}+h_2^*\eta_k(R_1,R_2)\right)\left(\eta_k(R_1,r)\sin\gamma_kt\right.\right.$$

$$\left.-\xi_k(R_1,r)\cos\gamma_kt\right)+\left(\psi_k^{(2)}+h_2^*\xi_k(R_1,R_2)\right)$$

$$\left.\times\left(\eta_k(R_1,r)\cos\gamma_kt+\xi_k(R_1,r)\sin\gamma_kt\right)\right]dtdr+\int_0^{\vartheta}h_2(t)S_2(t)(L_k^{(2)}\sin\gamma_kt-N_k^{(2)}\cos\gamma_kt)\,dt\Big\}.$$

For k = 0 one has

$$m_0^{(1)}=\frac{1}{W_0^{(1)}}\Big[-\ln\frac{R_2}{R_1}\sum_{j=1}^{\infty}j^{-\frac{5}{4}}(A_j^{(1)}m_j^{(1)}+D_j^{(1)}n_j^{(1)})+\frac{m_0^{(2)}}{R_1}+\frac{1}{\lambda R_1}\int_{R_1}^{R_2}\int_0^{\vartheta}rP(r,t)\ln\frac{R_2}{r}\,dtdr$$

$$+\ln\frac{R_2}{R_1}\int_0^{\vartheta}h_1(t)S_1(t)\,dt\Big], \tag{16}$$

$$m_0^{(2)}=\frac{1}{W_0^{(2)}}\Big[-\ln\frac{R_2}{R_1}\sum_{j=1}^{\infty}j^{-\frac{5}{4}}(A_j^{(2)}m_j^{(2)}+D_j^{(2)}n_j^{(2)})$$

$$+\frac{m_0^{(1)}}{R_2}+\frac{1}{\lambda R_2}\int_{R_1}^{R_2}\int_0^{\vartheta}rP(r,t)\ln\frac{r}{R_1}\,dtdr+\ln\frac{R_2}{R_1}\int_0^{\vartheta}h_2(t)S_2(t)\,dt\Big].$$

In this manner the totalities of the infinite systems (12), (14) and (16) have been obtained to determine $m_k^{(1)}$, $m_k^{(2)}$, $n_k^{(1)}$ and $n_k^{(2)}$. To investigate these systems an estimate is found first of the sum of the moduli of the coefficients of the unknowns $m_j^{(l)}$ and $n_j^{(l)}$ of each of the equations. If one bears in mind the previously assumed bounded variation of the functions $h_1(t)$ and $h_2(t)$ in the interval $(0, \vartheta)$ and uses an estimate for the Fourier coefficients of functions of bounded variation [3] as well as the Hölder inequality one obtains after some simplifications the following estimate for the sum $\sigma_k^{(1)}$ of the moduli of the coefficients of the unknowns in the k-th equation of the first system (12):

$$\sigma_k^{(1)}\leqslant\sqrt{\frac{2}{g_k^{(1)}}}\left[\frac{24H_1}{\pi}k^{-\frac{1}{4}}\sqrt{\eta_k^2(R_1,R_2)+\xi_k^2(R_1,R_2)}+\frac{H_2}{R_1}\right]. \tag{17}$$

In the above H_1 and H_2 are the total variations of the functions $h_1(t)$ and $h_2(t)$ in the interval $(0, \vartheta)$ respectively. Moreover, using an asymptotic representation for the Thomson functions [4] one obtains

$$\mathrm{ber}(x)=\frac{e^{\alpha(x)}}{\sqrt{2\pi x}}\cos\beta(x),\qquad \mathrm{bei}(x)=\frac{e^{\alpha(x)}}{\sqrt{2\pi x}}\sin\beta(x),$$
$$\mathrm{ker}(x)=\sqrt{\frac{\pi}{2x}}e^{\alpha(-x)}\cos\beta(-x),\quad \mathrm{kei}(x)=\sqrt{\frac{\pi}{2x}}e^{\alpha(-x)}\sin\beta(-x), \tag{18}$$

where

$$\alpha(x)\sim\frac{x}{\sqrt{2}}+\frac{1}{8x\sqrt{2}}-\frac{25}{384x^3\sqrt{2}}-\frac{13}{128x^4}-\dots,$$
$$\beta(x)\sim\frac{x}{\sqrt{2}}-\frac{\pi}{8}-\frac{1}{8x\sqrt{2}}-\frac{1}{16x^2}-\frac{25}{384x^3\sqrt{2}}+\dots, \tag{19}$$

and for $\mu_k > 1.25/(R_2-R_1)$ one obtains

$$\sigma_k^{(1)}<\frac{\sqrt{2}}{\sqrt{\mu_k^2+h_1^{*2}}}\left[\frac{24H_1}{\pi}k^{\frac{1}{4}}+H_2\sqrt{\frac{R_2}{R_1}}\exp\left(-\mu_k\frac{R_2-R_1}{\sqrt{2}}\right)\right]$$
$$<\sqrt{\frac{a\vartheta}{\pi}}\left[8H_1k^{-\frac{1}{4}}+\sqrt{\frac{R_2}{R_1}}H_2k^{-\frac{1}{2}}\exp\left(-\sqrt{k\pi}\frac{R_2-R_1}{\sqrt{a\vartheta}}\right)\right]. \tag{20}$$

One can see from (19) that $\sigma_k^{(1)}$ approaches zero at a rate of $O(k^{-1/4})$ and becomes less than unity starting with some k. Similar estimates for the sums of the moduli of the coefficients are obtained in the remaining systems (11) and (13). It can be seen from (13) and (15) that the free terms $p_k^{(l)}$ and $q_k^{(l)}$ remain bounded in their totality and also approach zero at a rate of $O(k^{-1/4})$. By the theory of infinite systems [5] the totality of the systems (12), (14) and (16) has a unique bounded solution and can be obtained by successive approximations.

By inserting into (4) the values of $M_k^{(1)}$ and $M_k^{(2)}$ given by (11) and grouping together in (6) the conjugate terms, one finally obtains for u(r, t)

$$u(r,t)=\frac{1}{\vartheta(\ln R_2-\ln R_1)}\left[\ln\frac{r}{R_1}\left(m_0^{(2)}+\frac{1}{\lambda}\int_r^{R_2}\int_0^{\vartheta}r_1P(r_1,t_1)\ln\frac{R_2}{r_1}dt_1dr_1\right)\right.$$
$$\left.+\ln\frac{R_2}{r}\left(m_0^{(1)}+\frac{1}{\lambda}\int_{R_1}^{r}\int_0^{\vartheta}r_1P(r_1,t_1)\ln\frac{r_1}{R_1}dt_1dr_1\right)\right]+\frac{2}{\vartheta}\sum_{k=1}^{\infty}\frac{1}{\eta_k^2(R_1,R_2)+\xi_k^2(R_1,R_2)}$$
$$\times\left\{\delta_k(r,R_2)\left[k^{-\frac{5}{4}}(m_k^{(1)}\cos\gamma_kt+n_k^{(1)}\sin\gamma_kt)-\frac{1}{\lambda}\int_{R_1}^{r}\int_0^{\vartheta}r_1P(r_1,t_1)\left(\eta_k(R_1,r_1)\cos\gamma_k(t_1-t)\right.\right.\right.$$
$$\left.\left.+\xi_k(R_1,r_1)\sin\gamma_k(t_1-t)\right)dt_1dr_1\right]+\tau_k(r,R_2)\left[k^{-\frac{5}{4}}(m_k^{(1)}\sin\gamma_kt-n_k^{(1)}\cos\gamma_kt)\right.$$
$$\left.+\frac{1}{\lambda}\int_{R_1}^{r}\int_0^{\vartheta}r_1P(r_1,t_1)\left(\eta_k(R_1,r_1)\sin\gamma_k(t_1-t)-\xi_k(R_1,r_1)\cos\gamma_k(t_1-t)\right)dt_1dr_1\right]$$
$$+\delta_k(R_1,r)\left[k^{-\frac{5}{4}}(m_k^{(2)}\cos\gamma_kt+n_k^{(2)}\sin\gamma_kt)-\frac{1}{\lambda}\int_r^{R_2}\int_0^{\vartheta}r_1P(r_1,t_1)\left(\eta_k(r_1,R_2)\cos\gamma_k(t_1-t)\right.\right.$$
$$\left.\left.+\xi_k(r_1,R_2)\sin\gamma_k(t_1-t)\right)dt_1dr_1\right]+\tau_k(R_1,r)\left[k^{-\frac{5}{4}}(m_k^{(2)}\sin\gamma_kt-n_k^{(2)}\cos\gamma_kt)\right.$$
$$\left.\left.+\frac{1}{\lambda}\int_r^{R_2}\int_0^{\vartheta}r_1P(r_1,t_1)\left(\eta_k(r_1,R_2)\sin\gamma_k(t_1-t)-\xi_k(r_1,R_2)\cos\gamma_k(t_1-t)\right)dt_1dr_1\right]\right\}, \tag{21}$$

where

$$\delta_k(x, y) = \eta_k(R_1, R_2)\eta_k(x, y) + \xi_k(R_1, R_2)\xi_k(x, y),$$
$$\tau_k(x, y) = \xi_k(R_1, R_2)\eta_k(x, y) + \eta_k(R_1, R_2)\xi_k(x, y). \tag{22}$$

It can be seen from (22), (11), (13), (18) and (19) that the terms of the series (21) decrease at the rate of $O(k^{-2} + k^{-5/4}e^{-\sqrt{2\mu}k(R_2-r)} + k^{-5/4}e^{-\sqrt{2\mu}k(r-R_1)})$. These estimates show that the series (21) together with its first derivatives is convergent in the region $R_1 < r < R_2$ and that at a point of discontinuity of the function $P(r, t)$ the function

$$\left[\frac{\partial}{\partial t} - \frac{a}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)\right]u(r, t)$$

converges to

$$\frac{1}{4c\rho}\left[P(r+0, t+0) + P(r+0, t-0) + P(r-0, t+0) + P(r-0, t-0)\right].$$

If the values of the functions $S_1(t)$, $S_2(t)$, $P(r, t)$ are given and also of the coefficients of heat exchange $h_1(t)$ and $h_2(t)$ as well as the ratios R_2/R_1 and $a\vartheta/R_1^2$, and if the systems (12), (14) and (16) are solved the estimates from above and from below are obtained for $m_k^{(l)}$ and $n_k^{(l)}$; subsequently, using the method described in [6] one can find the values of $u(r, t)$ both by deficiency and excess. It was shown in [7] that if the heat exchange coefficients $h_1(t)$, $h_2(t)$ are actually given as well as the functions $S_1(t)$, $S_2(t)$ and $P(r, t)$ one can considerably accelerate the rate of decrease in the coefficients determined from the infinite systems by a transformation of the unknowns $m_k^{(l)}$ and $n_k^{(l)}$; thus the number of operations required to obtain the specified in advance accuracy of the solution can be considerably reduced.

In conclusion some particular cases will be considered.

a) The cases are considered in which $h_1(t)$ and $h_2(t)$ are periodic piecewise linear discontinuous functions given by

$$h_l(t) = h_{l0} + h_{l1}\left[\frac{k_l(t-\vartheta_l)}{\vartheta} - E\left(k_l\frac{t-\vartheta_l}{\vartheta}\right)\right], \quad l = 1;\ 2, \tag{23}$$

where k_l are positive integers and $E(x)$ is the integral-part function. From (13) one obtains

$$A_0^{(l)} = 2h_{l0} + h_{l1}, \qquad A_k^{(l)} = 0, \qquad (k \geqslant 1), \qquad D_k^{(l)} = \begin{cases} 0 & \text{for } k \neq k_l j, \\ -\dfrac{h_{l1}}{j\pi} & \text{for } k = k_l j, \end{cases}$$

where j is a positive integer.

The case is now considered in more detail in which, for example, $k_1 = 1$, $h_2(t) = h_{20}$ = constant. It is assumed as regards $S_1(t)$ and $S_2(t)$ that they are continuous and have a derivative which is of bounded variation almost everywhere. It is also assumed that $P(r, t) = 0$. The system (12) is transformed by denoting

$$m_k^{(1)} = \frac{m_0^*}{T}\left[\nu k^{-\frac{1}{4}} + (2\nu^2\pi - \omega)k^{-\frac{3}{4}}\right] + \frac{m_k^*}{k},$$
$$m_0^{(1)} = S_1(0)\vartheta - m_0^* - 2\sum_{j=1}^{\infty} m_j^* j^{-\frac{9}{4}}, \tag{24}$$
$$n_k^{(1)} = -\frac{\nu}{T}m_0^* k^{-\frac{1}{4}} + \frac{n_k^*}{k},$$

where

$$\nu = \frac{h_{11}\sqrt{a\vartheta}}{4\pi^{\frac{3}{2}}}, \qquad \omega = \frac{h_{11}a\vartheta}{8\pi^2 R_1}(1 + 2h_{10}R_1 + h_{11}R_1),$$
$$T = 1 + 2\nu\sum_{j=1}^{\infty} j^{-\frac{3}{2}} - \frac{h_{11}a\vartheta}{24R_1}(1 + 2h_{10}R_1). \tag{25}$$

To determine the new unknowns the following system of linear equations is obtained after some transformations by inserting the values m_k^1 and n_k^1 from (21) into (12):

$$m_k^* = -\frac{h_{11}N_k^{(1)}}{\pi\, g_k^{(1)}}\, k^{\frac{5}{4}}\Big\{\frac{m_0^*}{T}\Big[\frac{1}{2} + \nu k^{-\frac{3}{2}} + \frac{7}{4k^2}(2\nu^2\pi - \omega) + \frac{g_k^{(1)}\pi}{h_{11}N_k^{(1)}}\Big(\nu k^{-\frac{1}{2}} + \frac{2\nu^2\pi - \omega}{k}\Big)$$

$$+ \frac{\nu}{N_k^{(1)}}\sum_{j=1}^{\infty}{}' \frac{jN_k^{(1)} + kL_k^{(1)}}{\sqrt{j}\,(k^2 - j^2)}\Big] + \sum_{j=1}^{\infty}{}' \frac{j^{-\frac{5}{4}}}{k^2 - j^2}\Big(\frac{kL_k^{(1)}n_j^*}{N_k^{(1)}} - jm_j^*\Big) + m_k^* k^{-\frac{9}{4}}$$

$$+ \frac{\pi k^{-\frac{5}{4}}}{h_{11}N_k^{(1)}R_1}\left[m_k^{(2)}\left(\psi_k^{(1)} + (2h_{10} + h_{11})\,\xi_k(R_1, R_2)\right) - n_k^{(2)}\left(\zeta_k^{(1)} + (2h_{10} + h_{11})\,\eta_k(R_1, R_2)\right)\right] \tag{26}$$

$$+ \frac{1}{2}\int_0^{\vartheta}\Big[S_1(t) + \Big(\frac{h_{10}\vartheta}{h_{11}} + t\Big)S_1'(t)\Big]\Big(\frac{L_k^{(1)}}{N_k^{(1)}}\sin\gamma_k t - \cos\gamma_k t\Big)\,dt\Big\},$$

$$n_k^* = -\frac{h_{11}L_k^{(1)}}{\pi g_k^{(1)}}\, k^{\frac{5}{4}}\Big\{\frac{m_0^*}{T}\Big[\frac{1}{2} + \nu k^{-\frac{3}{2}} + \frac{7}{4k^2}(2\nu^2\pi - \omega) - \frac{\nu\pi g_k^{(1)}}{h_{11}L_k^{(1)}}\, k^{-\frac{1}{2}} - \frac{\nu}{L_k^{(1)}}\sum_{j=1}^{\infty}{}' \frac{jL_k^{(1)} - kN_k^{(1)}}{\sqrt{j}\,(k^2 - j^2)}\Big] - \sum_{j=1}^{\infty}{}' \frac{j^{-\frac{5}{4}}}{k^2 - j^2}$$

$$\times\Big(jm_j^* + \frac{kN_k^{(1)}n_j^*}{L_k^{(1)}}\Big) + m_k^* k^{-\frac{9}{4}} + \frac{\pi k^{-\frac{5}{4}}}{h_{11}L_k^{(1)}R_1}\left[m_k^{(2)}\left(\psi_k^{(1)} + (2h_{10} + h_{11})\,\xi_k(R_1, R_2)\right) - n_k^{(2)}\left(\zeta_k^{(1)} + (2h_{10} + h_{11})\,\eta_k(R_1, R_2)\right)\right]$$

$$- \frac{1}{2}\int_0^{\vartheta}\Big[S_1(t) + \Big(\frac{h_{10}\vartheta}{h_{11}} + t\Big)S_1'(t)\Big]\Big(\cos\gamma_k t + \frac{N_k^{(1)}}{L_k^{(1)}}\sin\gamma_k t\Big)\,dt\Big\}.$$

If the asymptotic representation (18) and (19) for the Thomson functions is used one finds that the sums of the moduli of the coefficients in the k-th equation of the system (26) as well as the free terms decrease at the rate of $O(k^{-1/4})$.

As far as $m_k^{(2)}$ and $n_k^{(2)}$ are concerned the series appearing in Eq. (14) disappear in due time and $m_k^{(2)}$ and $n_k^{(2)}$ can be determined from the following expression:

$$m_k^{(2)} = \frac{1}{R_2 g_k^{(2)}}\Big\{-\Big[\frac{m_0^*}{T}\Big(\nu k^{-\frac{1}{4}} + (2\nu^2\pi - \omega)\,k^{-\frac{3}{4}}\Big) + \frac{m_k^*}{k}\Big]\left[\zeta_k^{(2)} + h_{20}\eta_k(R_1, R_2)\right]$$

$$- \Big(\frac{\nu}{T}m_0^* k^{-\frac{1}{4}} - \frac{n_k^*}{k}\Big)\left[\psi_k^{(2)} + h_{20}\xi_k(R_1, R_2)\right] + \frac{h_{20}\vartheta R_2}{2\pi}\, k^{\frac{1}{4}}\int_0^{\vartheta} S_2'(t)\left(N_k^{(2)}\cos\gamma_k t - L_k^{(2)}\sin\gamma_k t\right)dt\Big\},$$

$$n_k^{(2)} = \frac{1}{R_2 g_k^{(2)}}\Big\{\Big(\frac{\nu}{T}m_0^* k^{-\frac{1}{4}} - \frac{n_k^*}{k}\Big)\left[\zeta_k^{(2)} + h_{20}\eta_k(R_1, R_2)\right] \tag{27}$$

$$+ \Big[\frac{m_0^*}{T}\Big(\nu k^{-\frac{1}{4}} + (2\nu^2\pi - \omega)\,k^{-\frac{3}{4}}\Big) + \frac{m_k^*}{k}\Big]\left[\psi_k^{(2)} + h_{20}\xi_k(R_1, R_2)\right]$$

$$+ \frac{h_{20}\vartheta R_2}{2\pi}\, k^{\frac{1}{4}}\int_0^{\vartheta} S_2'(t)\left(L_k^{(2)}\cos\gamma_k t + N_k^{(2)}\sin\gamma_k t\right)dt\Big\}.$$

b) If $h_1(t)$ and $h_2(t)$ are piecewise constant: $h_l(t) = h_j^{(l)}$ for $t_j^{(l)} < t < t_{j+1}^{(l)}$ one obtains

$$A_k^{(l)} = \frac{2}{\gamma_k\vartheta}\sum_{j=1}^{\Omega_l}\left(h_{j-1}^{(l)} - h_j^{(l)}\right)\sin\gamma_k t_j^{(l)};$$

$$D_k^{(l)} = -\frac{2}{\gamma_k\vartheta}\sum_{j=1}^{\Omega_l}\left(h_{j-1}^{(l)} - h_j^{(l)}\right)\cos\gamma_k t_j^{(l)},$$

where Ω_l is the number of discontinuities of the function $h_l(t)$ in the interval $(0, \vartheta)$.

c) In the case in which the graph of the functions $h_1(t)$ and $h_2(t)$ represents a toothlike curve, "a fence":

$$h_l(t) = h_{l0} + h_{l1}\left[\frac{2k_l t}{\vartheta} - 2E\left(\frac{k_l t}{\vartheta}\right) - 1\right]\left[4E\left(\frac{k_l t}{\vartheta}\right) - 2E\left(\frac{2k_l t}{\vartheta}\right) + 1\right],$$

the Fourier coefficients of the functions $h_1(t)$ and $h_2(t)$ are given by

$$A_0^{(l)} = 2h_{l0} - h_{l1}, \qquad A_k^{(l)} = \begin{cases} 0 & \text{for} \quad k \neq (2j-1)\,k_l, \\ -\dfrac{2h_{l1}}{(2j-1)^2\,\pi^2} & \text{for} \quad k = (2j-1)\,k_l, \end{cases} \qquad D_k^{(l)} = 0.$$

d) For $h_1(t)$ = const and $h_2(t)$ = const the infinite systems of linear equations (12) and (14) degenerate into equalities and the solution is then identical with the one obtained by using classical methods.

LITERATURE CITED

1. A. V. Lykov, Theory of Heat Conduction, Izd. Vysshaya Shkola, Moscow (1967).
2. K. J. Tranter, Integral Transforms of Mathematical Physics, [Russian translation], Gostekhizdat, Moscow (1956).
3. A. Zigmund, Trigonometric Series [Russian translation], Vol. 1, Mir, Moscow (1965).
4. G. N. Watson, Theory of Bessel Functions [Russian translation], IL, Moscow (1949).
5. L. V. Kantorovich and V. I. Krylov, Approximation Methods of Higher Analysis, Fizmatgiz, Moscow–Leningrad (1962).
6. R. S. Minasyan, Izv. AN ArmSSR, Ser. Fiz.-Mat. Nauk, 11, 41-62 (1958).
7. R. S. Minasyan, in: Heat and Mass Transfer, No. 8, Minsk (1968), p. 205.